Revisiting stock market index correlations

October 5, 2007

Preliminary and incomplete

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Abstract

Comovement of stock market indices increases during turmoil, and does not come down when the turmoil settles down. This paper explain this upgrade of comovements during turmoil periods with theories from Bayesian learning and dynamical systems involving synchronization.

Keywords: output correlations, business cycles, synchronization, stock market, stochastic systems.

1 Introduction

Forbes and Rigobon (2002) report that during times of high volatility in the series under consideration correlation coefficient between these series is biased upwards. Lee and Kim (1993) find that average weekly cross market correlations between 12 major stock markets increase from 0.23 before the October 1987 crash in the U.S. markets to 0.39 afterwards. Forbes and Rigobon (2002) explain why the correlations get high during the crash, but they do not explain the phenomena observed by Lee and Kim (1993). Why the correlations stay high after the crash? This paper is an attempt to explain the stickiness in correlation coefficients between stock markets by referring to theory of Bayesian learning.

Our main conclusion is that the correlation does not go down because it is learned during the turmoil. This learned level of correlation has high precision, so there is little doubt that it is at a higher level because of a numerical discrepancy. The belief that market movements are loyal to each other turns into a self-fulfilling prophecy. Traders follow other markets closely before making trading decisions. So the belief that interdependence between markets are high during the crash turns into reality by correlated actions of traders in different markets avoiding correlation to fall to its previous level after the crash.

Is there a draw back in high level of synchronization between markets over the world? One concern may be diminishing opportunities of cross-country hedging when markets start to move up and down altogether. Cross-country hedging may be of secondary importance after cross-industry hedging when the world economies are synchronizing through trade. Also, the correlation coefficient measuring comovements at higher frequencies may fail to measure the secular growth in the indices. So hedging opportunities still may exist for investors targeting long term growth rather that short term speculations.

2. Bayesian Learning

This discussion closely follows the extensive review by Chamley (2004). The information structure of the model is as follows:

1. The value of the nature's parameter θ is chosen randomly before the first period according to a normal distribution $N(\bar{\theta}, \sigma_\theta^2)$.
2. There is a countable number n of individuals that receive a private signal s_i, $i = 1, \dots, n$: $s_i = \theta + \epsilon_i$. $Corr(\epsilon_i, \epsilon_j) = 0$ for $i \neq j$, and $\epsilon \sim N(0, \sigma_\epsilon^2)$. All individuals have the same payoff function from their actions a: $U(a) = -E[(a-\theta)^2]$. Individual t chooses her action $a_t \in \mathbb{R}$ once and for all in period t. Each individual chooses an action only once, and at each time period, there is only one individual acting. The individual i is assigned to time t exogenously.
3. The public information at the beginning of period t is made of the prior distribution $N(\bar{\theta}, \sigma_\theta^2)$ and the set of previous actions $H_t = \{a_{t-1}, \dots, a_1\}$.

2.1 Perfect observation of actions

The common quadratic payoff function induces the individuals to choose their beliefs $\tilde{\mu}_t \triangleq E_t[\theta]$ as their action: $a_t = \tilde{\mu}_t$ for all t. Since all individuals know the common payoff function, and aware that it is at the same time common information, they are aware that the actions reflect beliefs perfectly. So the public belief one period later, μ_{t+1} is equal to the individual t's personal belief: $\mu_{t+1} = \tilde{\mu}_t$. The public belief $\beta_t \sim N(\mu_t, \sigma_t^2)$ is updated according to the Bayesian rule:

$$\begin{aligned} \mu_{t+1} &= (1-\alpha_t)\mu_t + \alpha_t s_t, && (1) \\ \rho_{t+1} &= \rho_t + \rho_\epsilon. && (2) \end{aligned}$$

where $\alpha_t = \frac{\rho_\epsilon}{\rho_\epsilon + \rho_t}$, and $\rho_z \triangleq \sigma_z^{-2}$ is the precision of a random variable Z in general. As shorthand, $\mu_t \triangleq \mu_{\beta_t}$ and $\rho_t \triangleq \rho_{\beta_t}$ are used.

The updating of mean and variance in (1) and in (2) of the public belief is referred to as "Gaussian learning rule" and is a consequence of the normal distribution being the *conjugate family* of itself. [1] The Gaussian updating rule is particularly interesting since the updating of the precision ρ is independent of the mean μ.

In the linear case, the recursive equation (2) could be identified as a function of time:

$$\rho_{t+1} = \rho_\theta + t\rho_\epsilon. \tag{3}$$

So precision grows indefinitely in time, and public belief becomes "almost certain" as $t \to \infty$. The variance $\sigma_t^2 = 1/\rho_t$ converges to zero like $1/t$.

2.2 The circular case

The counterpart of the normal distribution on the circle is the von Mises distribution which has some desirable properties similar to the normal distribution. These properties are given by Mardia and Jupp (2000, pp.41-43). In this study, one additional property of the von Mises distribution is crucial: like the normal distribution, it is the conjugate family of itself. That is, if the prior (distribution of β_t) and the signal x_t are von Mises, then the posterior (distribution of β_{t+1}) is von Mises as well.

The setup of the model is similar to the linear case:

1. The value of the nature's parameter μ is chosen randomly before the first period according to a von Mises distribution $M(\delta, \kappa_\mu)$, where δ is the *mean direction* and κ_μ is the precision.

[1] That is, if the signal s_t is normal, then the set of possible common distributions for the prior β_t and the posterior β_{t+1} form the conjugate family of the normal distribution.

2. There is a countable number n of individuals that receive a private signal x_i, where the stochastic process $\{x_i\}_{i=1,\ldots,n}$ is defined on a probability space $(\Omega, \mathcal{F}, P)$ where $\mathcal{F}$ is a σ-algebra on the set Ω, and P is a probability measure on the measurable space $(\Omega, \mathcal{F})$. Specifically, $x_i = \mu + \epsilon_i$. $Corr(\epsilon_i, \epsilon_j) = 0$ for $i \neq j$, and $\epsilon \sim M(0, \kappa)$. All individuals have the same payoff function from their actions a: $U(a) = -E[(a-\mu)^2]$. Individual t chooses his action $a_t \in \mathbb{R}$ once and for all in period t. Each individual chooses an action only once, and at each time period, there is only one individual acting. The individual i is assigned to time t exogenously.

3. The public information at the beginning of period t is made of the prior distribution $M(\delta, \kappa_\mu)$ and the set of previous actions $H_t = \{a_{t-1}, \ldots, a_1\}$.

2.3 Perfect observation of actions

The common payoff function induces the individuals to choose their beliefs $\tilde{\mu}_t \triangleq E_t[\mu]$ as their action: $a_t = \tilde{\mu}_t$ for all t. Since all individuals know the payoff function is common, they are aware that the actions reflect beliefs perfectly. So the public belief one period later, μ_{t+1} is equal to the individual t's personal belief: $\mu_{t+1} = \tilde{\mu}_t$. The public belief $\beta_t \sim M(\mu_t, \kappa_t)$ is updated according to the Bayesian rule below (Bagchi, 1994) to become the public belief for the next period: $\beta_{t+1} \sim M(\mu_{t+1}, \kappa_{t+1})$.

$$\mu_{t+1} = \tan^{-1} \frac{\kappa \sin x_t + \kappa_t \sin \mu_t}{\kappa \cos x_t + \kappa_t \cos \mu_t}, \quad (4)$$

$$\kappa_{t+1}^2 = \kappa_t^2 + \kappa^2 + 2\kappa\kappa_t \cos(\mu_t - x_t). \quad (5)$$

The updating rule in the von Mises case is particularly interesting: unlike the normal case, the updating rule for precision depends on the signal x_t and the prior mean μ_t. Three extreme cases will be considered:

1. *Assertive signal*
   When $x_t = \mu_t$ for a certain time interval t, then κ_t grows by the precision of the signal: $\kappa_{t+1} = \kappa_t + \kappa$. This case is identical with the Gaussian case.

2. *Orthogonal signal*
   When $|\mu_t - x_t| = \pi/2$ at a certain time interval t, then κ_t^2 grows by κ^2: $\kappa_{t+1}^2 = \kappa_t^2 + \kappa^2 \Rightarrow \kappa_{t+1} < \kappa_t + \kappa$.

3. *Negative signal*
   When $|\mu_t - x_t| = \pi$ at a certain time interval t, then κ_t may increase or decrease: $\kappa_{t+1} = |\kappa_t - \kappa|$.

In general, $\lim_{t\to\infty} P(A(\kappa)\kappa t < \kappa_{t+1} < \kappa t) = 1$.[2] If the incoming signal has any vector

[2]see proof of Lemma 1. $0 < A(\kappa) \triangleq I_1(\kappa)/I_o(\kappa) < 1$, for $0 < \kappa < \infty$ where $I_q(\cdot)$ is the modified Bessel function of the first kind of order q. $A(\kappa)$ is increasing.

component in the direction of the common belief, definitely there will be gains in terms of precision. Specifically, the following proposition applies:

Proposition 1 *$\kappa_{t+1} > \kappa_t$ when $\cos(\mu_t - x_t) > -\frac{\kappa}{2\kappa_t}$. If κ_t is growing, history is gaining more importance over the private signal through formula (4).*

Next, we will propose and prove a lemma showing that κ_{t+1}/κ converges to $tA(\kappa)$ in distribution.

Lemma 1 *$\lim_{t\to\infty} P(\kappa_{t+1}/\kappa > tA(\kappa)) = 1$.*

Proof.

From (4) and (5) we have the recursive relation

$$z_{t+1} = z_t + u_t \Rightarrow z_{t+1} = z_o + \sum_{j=1}^{t} u_j. \tag{6}$$

for z_t and u_t complex where $z_t \equiv \kappa_t \cos\mu_t + i\kappa_t \sin\mu_t$, $u_t \equiv \kappa\cos x_t + i\kappa \sin x_t$, and $z_o \equiv \kappa_\mu \cos\delta + i\kappa_\mu \sin\delta$.

Then we obtain the real recursive equations

$$\kappa_{t+1}\cos\mu_{t+1} = \kappa_\mu \cos\delta + \kappa\sum_{j=1}^{t}\cos x_j \quad\Rightarrow\quad C_t \triangleq \sum_{j=1}^{t}\cos x_j = \frac{\kappa_{t+1}}{\kappa}\cos\mu_{t+1} - \frac{\kappa_\mu}{\kappa}\cos\delta, \tag{7}$$

$$\kappa_{t+1}\sin\mu_{t+1} = \kappa_\mu \sin\delta + \kappa\sum_{j=1}^{t}\sin x_j \quad\Rightarrow\quad S_t \triangleq \sum_{j=1}^{t}\sin x_j = \frac{\kappa_{t+1}}{\kappa}\sin\mu_{t+1} - \frac{\kappa_\mu}{\kappa}\sin\delta. \tag{8}$$

C_t and S_t are the sufficient statistics that represent the history H_t similar to the sufficient statistic $T_t \triangleq \sum_{j=1}^{t} s_j$ in the linear case. Mardia and Jupp (2000) give the distributions for $R_t \equiv \sqrt{C_t^2 + S_t^2}$, and $\bar{\theta}_t \equiv \tan^{-1}(S_t/C_t)$. We will establish divergence of κ_{t+1}/κ without using the distribution function explicitly. We use (7) and (8) and $R_t^2 = C_t^2 + S_t^2$ to derive an expression for $\kappa^2 R_t^2$:

$$\kappa^2 R_t^2 = \kappa_{t+1}^2 + \kappa_\mu^2 - 2\kappa_{t+1}\kappa_\mu \cos(\mu_{t+1} - \delta). \tag{9}$$

From Mardia and Jupp (2000, p.75)

$$E[\kappa^2 R_t^2] = \kappa^2 t^2 \rho^2 + \kappa^2 t(1-\rho^2) = \kappa_\mu^2 + E[\kappa_{t+1}^2] - 2\kappa_\mu E[\kappa_{t+1}\cos(\mu_{t+1} - \delta)] \tag{10}$$

where ρ is the mean resultant length of the distribution under consideration. Specifically for the von Mises case, $\rho \equiv A(\kappa)$.

Collecting terms and dividing by t^2, we get

$$E[\kappa_{t+1}^2/t^2] = \kappa^2\rho^2 + \frac{\kappa^2}{t}[1-\rho^2] - \frac{\kappa_\mu^2}{t^2} + 2\kappa_\mu E[\frac{1}{t^2}\kappa_{t+1}\cos(\mu_{t+1}-\delta)]. \tag{11}$$

All the terms except the first vanishes as $t \to \infty$. The last term vanishes since from the case of an assertive signal we know $\sup_\omega \kappa_{t+1}(\omega) = \kappa_\mu + \kappa t$, $\forall \omega \in \Omega$ and thus $\lim_{t\to\infty} \kappa_{t+1}/t^2 = 0$ with certainty. Since the cosine function is bounded, it is also true that $\lim_{t\to\infty} \kappa_{t+1}\cos(\mu_{t+1}-\delta)/t^2 = 0$ with certainty. We have established that $\lim_{t\to\infty} E[\kappa_{t+1}^2/t^2] = \kappa^2\rho^2$, or

$$\lim_{t\to\infty} E\left|\frac{\kappa_{t+1}^2}{\kappa^2 t^2} - \rho^2\right| = 0 \tag{12}$$

which implies [3]

$$\frac{\kappa_{t+1}^2}{\kappa^2 t^2} \to_{L_1} \rho^2 \Rightarrow \frac{\kappa_{t+1}^2}{\kappa^2 t^2} \to_p \rho^2 \Rightarrow \frac{\kappa_{t+1}^2}{\kappa^2 t^2} \to_d \rho^2. \tag{13}$$

Since the square root function is measurable and continuous over the domain $[0,\infty)$, we also have $\frac{\kappa_{t+1}}{\kappa t} \to_d \rho$. Since $\frac{\kappa_{t+1}}{\kappa t}$ converges to a constant in distribution, we may conclude that $\lim_{t\to\infty} P\left(\frac{\kappa_{t+1}}{\kappa t} > \rho\right) = 1 \Rightarrow \lim_{t\to\infty} P\left(\frac{\kappa_{t+1}}{\kappa} > t\rho\right) = 1$. $\square$

Proposition 2 *In a circular social belief system with updating rules* (4) *and* (5)*, in determining agents' actions, the probability that the weight of public belief (relative to private signals) tending to infinity becomes equal to one as* $t \to \infty$*.*

Proof. This is a direct indication of Lemma 1.

Remark 1 *Notice that proposition* 2 *is for a belief distributed with some circular distribution updated by* (4) *and* (5)*. It is valid for, but not limited to beliefs distributed with a von Mises distribution.*

Remark 2 *Mardia and Jupp (2000) also show that the conditional limiting distribution of* μ_{t+1} *when the belief has a von Mises distribution is also von Mises:* $\mu_{t+1}|\kappa_{t+1} \sim M(\mu, \kappa_{t+1})$ *so that* $E[\mu_{t+1}|\kappa_{t+1}] = \mu$ *as* $t \to \infty$*. The expected value of the public belief, given the value of precision of the belief, converges to the true value* μ *as time goes to infinity, and with* κ_{t+1} *tending to infinity in the sense of Lemma 1.*

2.4 Learning a leading indicator

In this section, a two dimensional circular learning system will be investigated. The support of the distribution in this case will be again a manifold, a torus. It will be assumed

[3] The converge concepts here are $\to_{L_1}$: convergence in the first moment, $\to_p$: convergence in probability, and $\to_d$: convergence in distribution.

that the expected cosine of the difference between signal x_{1t} and x_{2t} is the cosine of a certain angle α. For this case, Mardia and Jupp (2000) propose a distribution on a torus with density proportional to:

$$\exp\{\kappa_1 \cos(x_{1t} - \mu_1) + \kappa_2 \cos(x_{2t} - \mu_2) + (\cos x_{1t}, \sin x_{1t})\mathbf{D}(\cos x_{2t}, \sin x_{2t})'\} \tag{14}$$

where $\mathbf{D} = a \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$ which is a multiple of a rotation matrix.

Also, we used the notation $(\cdot,\cdot)'$ to denote transpose of a vector.

This setting is similar to the one dimensional setting discussed before. Nature picks two variables μ_1 and μ_2, the constant change (frequency) in two economic circular variables at each time interval, from a distribution proportional to

$$\exp\{\kappa_{\mu_1} \cos(\mu_1 - \delta_{\mu_1}) + \kappa_{\mu_2} \cos(\mu_2 - \delta_{\mu_2}) + (\cos\mu_1, \sin\mu_1)\mathbf{D}(\cos\mu_2, \sin\mu_2)'\} \tag{15}$$

where $\mathbf{D}$ is as defined above.

The parameters of the distribution, $\kappa_{\mu_1}, \delta_{\mu_1}, \kappa_{\mu_2}, \delta_{\mu_2}$ and $\mathbf{D}$ are known. Each time interval has two sub-intervals, t_- and t_+. Signal x_{1t} arrives at the beginning of t_- (or t) and x_{2t} arrives at the beginning of t_+. Since the signals are correlated through matrix $\mathbf{D}$, agents will use x_{1t} as a "leading indicator" to form their expectations on x_{2t}. The Bayesian updating rules for this case will be

$$\begin{aligned} \mu_{jt+1} &= \tan^{-1}\frac{\kappa_{jt}\cos\mu_{jt} + \kappa_j \cos x_{jt}}{\kappa_{jt}\sin\mu_{jt} + \kappa_j \sin x_{jt}}, \\ \kappa^2_{jt+1} &= \kappa^2_{jt} + \kappa^2_j + 2\kappa_j\kappa_{jt}\cos(x_{jt} - \mu_{jt}) \end{aligned} \tag{16}$$

for $j = 1, 2$. Association between the signals will be learned according to the updating rules below:

$$\begin{aligned} \triangle\psi_{t+1} &= \tan^{-1}\frac{R_t \cos\triangle\psi_t + a\cos(x_{1t} - x_{2t})}{R_t \sin\triangle\psi_t + a\sin(x_{1t} - x_{2t})}, \\ R^2_{t+1} &= R^2_t + a^2 + 2aR_t\cos(x_{1t} - x_{2t} - \triangle\psi_t) \end{aligned} \tag{17}$$

where the belief on $\cos\alpha$ has mean $\cos\triangle\psi_t$ and precision R_t at the end of period t. The agents can use the identities $E_{t_-}\cos(x_{1t} - x_{2t}) = \rho_t \cos\triangle\psi_t$ and $E_{t_-}\sin(x_{1t} - x_{2t}) = \rho_t \sin\triangle\psi_t$ to derive

$$E_{t_-}\cos(x_{2t} - \mu_{2t}) = \rho_t \cos(x_{1t} - \mu_{2t} - \triangle\psi_t). \tag{18}$$

The unknown parameter ρ_t could be approximated as $\rho_t \simeq A(R_t + A^{-1}(A(\kappa_{1t})A(\kappa_{2t})))$ under the assumption that $\alpha = \mu_1 - \mu_2$. Note that ρ_t is increasing in R_t, κ_{1t}, and κ_{2t}.

The scenario is as follows: the index of market 1 makes a big jump down as a result of a local shock. The investors in market 2 forecast their market index using market 1 index as a leading indicator according to the formula above. Their forecast indicate a fall in their index as there is a learned association between two indices, and also due to the fact that now there is a big change in the usual frequency of market 1. Investors in market 1 start to sell in order not to get hit by the forecasted downfall.

The accuracy of the association R_t may fall at the beginning, but as both indices start falling fast, R_t increase with repetitive observations arriving confirming similar changes in the phases of the indices. With increasing R_t investors start to ignore the new signals, but go with their belief that the indices have a strong association between them. The strong association between the indices turns into a self-fulfillig prophecy.

Even when small local shocks start to take effect again, the learned strong association between the indices prevails. It takes a long time before the investors de-learn the strong association between the indices, and for the correlation between the indices to go down to the level before the shock.

3 Dynamical Systems Approach

3.1 Synchronization of indices

It is known that two systems linked with "coupling parameters" δ_1 and δ_2, and "natural frequencies" ω_1 and ω_2 will synchronize, and there will be a steady-state equilibrium where the phase difference between the series is constant over time. An unstable state is also present where the phase difference will be fluctuating, generating a quasiperiodic state in time domain (Stratonovich, 1967, chapter 9, sec.2).

Definition 3.1 *Two coupled systems are defined, in general, as*

$$\begin{aligned} \dot{X}_t &= f_1(X_t) - \delta_1 P_1(X_t, Y_t), \\ \dot{Y}_t &= f_2(Y_t) + \delta_2 P_2(X_t, Y_t), \quad \delta_1 > 0,\ \delta_2 > 0. \end{aligned} \tag{19}$$

where P_1 and P_2 are deterministic functions. When the functions f_1 and f_2 are assumed to be generating series with constant amplitudes, and with noisy frequencies averaging to ω_1 and ω_2 respectively (i.e., the processes $\{X_t\}$ and $\{Y_t\}$ go through a steady state cycle when independent) the analysis could be carried out in terms of phase series only.

$$\begin{aligned} \dot{\phi_{1t}} &= \omega_1 - \delta_1 Q_1(\phi_{1t}, \phi_{2t}) + \zeta_{1t}, \\ \dot{\phi_{2t}} &= \omega_2 + \delta_2 Q_2(\phi_{1t}, \phi_{2t}) + \zeta_{2t}, \end{aligned} \tag{20}$$

where ζ_{1t} and ζ_{2t} are additive noise terms on observed frequencies of the series. The functions Q_1 and Q_2 are the phase space counterparts of functions P_1 and P_2 respectively, derived under the assumption of constant amplitudes. For details on phase space representation and approximation of functions, see Aronson, Ermentrout, Kopell (1990) and Pikovsky et. al. (2001, pp.222-229).

Assuming that synchronization occurs in only one frequency of the series, and omitting all the resonance terms, the system can be represented in terms of the phase difference $\psi_t \equiv \phi_{1t} - \phi_{2t}$ between the series.

$$\dot{\psi}_t = \nu - (\delta_2 + \delta_1) \sin \psi_t + \zeta_{1t} - \zeta_{2t}, \tag{21}$$

with a first degree Fourier expansion of functions Q_1 and Q_2 that are assumed to be identical. Also $\nu \equiv \omega_1 - \omega_2$.

Stratonovich (1967) gives the solution for the stationary distribution of ψ under bounded noise[4], and for $\delta_2 + \delta_1 > 0$. When $\nu = 0$, this distribution is von Mises, with $\psi \sim M(0, \kappa)$, $\kappa \triangleq (\delta_2 + \delta_1)/\sigma^2$ where $\sigma^2 \triangleq Var(\zeta_1 - \zeta_2) \equiv \sigma_1^2 + \sigma_2^2 - 2\sigma_{12}$, $\sigma_j^2 \triangleq Var\zeta_j$, $j = 1, 2$, $\sigma_{12} \triangleq Cov(\zeta_1, \zeta_2)$. The correlation coefficient r in this case will be an increasing function of κ as derived by Koopmans (1974):

$$r \equiv A(\kappa) \tag{22}$$

where $A(\kappa) \equiv I_1(\kappa)/I_0(\kappa)$ and I_p is the modified Bessel function of the first kind of order p . Since $A(\kappa)$ is monotonically increasing, this approach implies increasing correlation between the series when covariance between the noise terms is increasing. Also when $\triangle\delta_2 + \triangle\delta_1 > 0$, the correlation between the series will be increasing. This feature is similar to features of the models introduced by Barnett and Dalkir (2007, equation 3.34) and Dalkir (2004, equation 8).

The case where $\nu \neq 0$ is slightly more complicated. In this situation the mean of the stationary distribution is not zero. Under bounded noise the phase difference will be stable around the steady state solution $\psi^\star \neq 0$.[5] If we define $\gamma_\psi \triangleq \dot{\psi}$, then the time average of γ_ψ will be

$$\overline{\gamma_\psi} \equiv \pi^{-1}\sigma^2 |I_{i\bar{\nu}}(\kappa)|^{-2} \sinh(\pi\bar{\nu}) \tag{23}$$

where $\bar{\nu} \equiv \nu/\sigma^2$ and κ is as defined above. Also $I_{i\bar{\nu}}(\cdot)$ is the modified Bessel function of the first kind of complex order.

The dynamical system approach is also in parallel with the Bayesian learning approach of section 2. When the index of market 1 starts falling as a result of a shock, this will be observed as a common shock by the investors of market 2. In the system above, this will be reflected as an increase in the covariance σ_{12} believed to exist between the indices. As the formula (23) (or formula (22)) implies, an increase in covariance will initiate a drop in average change in phase difference (Stratonovich 1967, pg.242) (or increase in correlation between the series). As the phase differences converge (correlation increase), the value of δ_2, representing association between indices 2 and 1 is believed to increase. That will further decrease the average change in the phase difference $\overline{\gamma_\psi}$ (increase the correlation coefficient r). In the case where investors believe that the effect of the common shock vanished, or σ_{12}

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[4]Namely, if $|\zeta_1 - \zeta_2| < \pi$.

[5]If $|\zeta_1 - \zeta_2| < \pi - 2\psi^\star$ where $\psi^\star = \sin^{-1} \frac{\nu}{\delta_1 + \delta_2}$, $-\frac{\pi}{2} < \psi^\star < \frac{\pi}{2}$ is the steady state phase difference. Also $\nu \neq 0 \Rightarrow \psi^\star \neq 0$.

drops back to its previous level before the shock, the persisting higher level of $\delta_1 + \delta_2$ will keep $\overline{\gamma_\psi}$ at its lower level (r at its higher level).

3.2 Reduction in volatility

An immediate question follows after the previous section is whether the stronger association between stock market indices have any advantages. One advantage that could be cited under the dynamical systems framework is the reduction in the volatility of the systems when there is an association between them.

Malakhov (1968) provides the formal treatment of the situation. If the stock market indices are closely associated so that we can neglect the volatility in their phase difference relative to the volatility in the individual phase series, the condition for reduced volatility compared to the case where the indices have no association between them becomes as the following.

Proposition 3 *If two stock markets are close to a synchronized state, that is the volatility in their phase difference ψ is negligible relative to the volatility in the individual phase series, σ_1^2 and σ_2^2, the variance of both index series will be at the same level $(\delta_2^2\sigma_1^2 + \delta_1^2\sigma_2^2)/(\delta_1+\delta_2)^2$ and further, if*

$$\frac{\delta_2}{2\delta_1 + \delta_2} < \frac{\sigma_2^2}{\sigma_1^2} < \frac{\delta_1 + 2\delta_2}{\delta_1} \tag{24}$$

holds then this common variance will be smaller than or equal to the variances of the independent indices, σ_1^2 and σ_2^2.

Proof.

The proof closely follows Malakhov (1968) as translated in Pikovsky *et al.* (2001.) First define $\psi = \phi_1 - \phi_2$, $\theta = \delta_2\phi_1 + \delta_1\phi_2$, $\nu = \omega_1 - \omega_2$. Then

$$\begin{aligned} \dot{\psi} &= \nu - (\delta_1+\delta_2)\sin\psi + \zeta_1 - \zeta_2, \\ \dot{\theta} &= \delta_2\omega_1 + \delta_1\omega_2 + \delta_2\zeta_1 + \delta_1\zeta_2, \\ Var(\dot{\theta}) &= \delta_2^2\sigma_1^2 + \delta_1^2\sigma_2^2. \end{aligned} \tag{25}$$

Write the phases in the form $\phi_1 = \frac{\delta_1\psi+\theta}{\delta_1+\delta_2}$ and $\phi_2 = \frac{-\delta_2\psi+\theta}{\delta_1+\delta_2}$. When the economies are phase locked, their phase difference is constant, so that $\dot{\psi} = 0$. We find

$$Var(\dot{\phi}_1) = \frac{\delta_2^2\sigma_1^2 + \delta_1^2\sigma_2^2}{(\delta_1+\delta_2)^2} = Var(\dot{\phi}_2). \tag{26}$$

For the closed economies, $\widetilde{Var}(\dot{\phi}_j)$ is σ_j^2, $j = 1, 2$. It is easy to verify that

$$\frac{\delta_2^2\sigma_1^2 + \delta_1^2\sigma_2^2}{(\delta_1+\delta_2)^2} < \sigma_j^2, \quad j = 1, 2, \tag{27}$$

when the proposition is satisfied. □

The proposition 3 could specifically be considered for equally strong association between the markets: $\delta_1 = \delta_2$. In this case, the condition in proposition 3 will be reduced to $1/3 < \sigma_2^2/\sigma_1^2 < 3$. Both stock markets will benefit from a reduction in volatility in this case, and their common volatility will be $(\sigma_1^2 + \sigma_2^2)/4$. Additionally, if these two markets have equal volatility when they are not associated $(\sigma_1^2 = \sigma_2^2)$, then their volatility will reduce by half once they become synchronized.

4 Conclusion

Our main conclusion is that the correlation does not go down because it is learned during the turmoil. This learned level of correlation has high precision, so there is little doubt that it is at a higher level because of a numerical discrepancy. The belief that market movements are loyal to each other turns into a self-fulfilling prophecy after the crisis. Traders follow other markets closely before making trading decisions. It is the belief that interdependence between markets are high during the crash that avoids correlation to fall to its previous level after the crash.

A question is due: Are comovements more synchronized after crashes in all frequency bands?

References

Aronson, D.G., Ermentrout, G.B., and Kopell, N., (1990) "Amplitude response of coupled oscillators," *Physica D*, vol.41, pp.403-449.

Barnett, W. A., and Dalkir, M., (2006) "Gains from synchronization," *Studies in Nonlinear Dynamics and Econometrics*, vol.11, issue 1, Berkeley Press.

Baxter, M., and Kouparitsas, M. A., (2005) "Determinants of business cycle comovement: A Robust analysis," *Journal of Monetary Economics,* vol.52, issue 1, pp.113-157.

Dalkir, M., (2004) "A new approach to causality in the frequency domain," *Economics Bulletin,* **3** 44, pp. 1-14.

Forbes, K. J., and Rigobon, R., (2002) "No contagion, only interdependence: measuring stock market comovements," *The Journal of Finance,* vol.57, issue 5.

Jennrich, R. I., (1970) "An asymptotic χ^2 test for the equality of two correlation matrices," *Journal of the American Statistical Association,* vol.35, no.330, pp.904-912.

Koopmans, L. H., (1974) *The spectral analysis of time series*, Academic Press.

Lee, S. B., and Kim, K. J., (1993) "Does the October 1987 crash strengthen the comovements among national stock markets?" *Review of Financial Economics,* vol.3, pp.89-102.

Lumsdaine, R. L., and Prasad, E. S., (2003) "Identifying the common component of international economic fluctuations: a new approach," *The Economic Journal,* vol.113, pp.101-127.

Mardia, K. V., and Jupp, P. E., (2000) *Directional statistics*, John wiley and Sons, London.

Nakagawa, N., and Kuramoto, Y. (1993) "Collective chaos in a population of globally coupled oscillators," *Prog. Theor. Phys.,* vol.89, issue 2.

Pikovsky, A., Rosenblum, M., and Kurths, J., (2001) *Synchronization,* Cambridge University Press.

Stratonovich, R. L., (1967) *Topics in the theory of random noise*, Vol. 2, Gordon and Breach Science Publishers, Inc., New York.